

Fermionic representations for characters of $\mathcal{M}(3, t)$, $\mathcal{M}(4, 5)$, $\mathcal{M}(5, 6)$ and $\mathcal{M}(6, 7)$ minimal models and related dilogarithm and Rogers–Ramanujan-type identities*

Andrei G Bytsko

Steklov Mathematics Institute, Fontanka 27, St Petersburg 191 011, Russia

Received 23 April 1999

Abstract. Characters and linear combinations of characters that admit a fermionic sum representation as well as a factorized form are considered for some minimal Virasoro models. As a consequence, various Rogers–Ramanujan-type identities are obtained. Dilogarithm identities producing corresponding effective central charges and secondary effective central charges are derived. Several ways of constructing more general fermionic representations are discussed.

1. Introduction

A minimal Virasoro model [1] $\mathcal{M}(s, t)$ is parametrized by two positive integers s and t such that $\langle s, t \rangle = 1$ (i.e. they are co-prime numbers). It has the central charge $c(s, t) = 1 - \frac{6(s-t)^2}{st}$. The characters of its irreducible representations with highest weights $h_{n,m}^{s,t} = \frac{(nt-ms)^2 - (s-t)^2}{4st}$ are given by [2]

$$\chi_{n,m}^{s,t}(q) = \frac{q^{\eta_{n,m}^{s,t}}}{(q)_{\infty}} \sum_{k=-\infty}^{\infty} q^{stk^2} (q^{k(nt-ms)} - q^{k(nt+ms)+nm}) \tag{1.1}$$

where $1 \leq n \leq s - 1$, $1 \leq m \leq t - 1$, $\eta_{n,m}^{s,t} := h_{n,m}^{s,t} - \frac{c(s,t)}{24}$ and $(q)_m := \prod_{k=1}^m (1 - q^k)$. The characters possess the following symmetries:

$$\chi_{n,m}^{s,t}(q) = \chi_{m,n}^{t,s}(q) = \chi_{s-n,t-m}^{s,t}(q) = \chi_{t-m,s-n}^{t,s}(q). \tag{1.2}$$

In addition, (1.1) allows us to relate some characters of different models

$$\chi_{\alpha n,m}^{\alpha s,t}(q) = \chi_{n,\alpha m}^{s,\alpha t}(q) \tag{1.3}$$

where α is a positive number such that $\langle s, \alpha t \rangle = \langle t, \alpha s \rangle = 1$. For instance, $\chi_{m,2}^{5,6}(q) = \chi_{1,2m}^{3,10}(q)$, $m = 1, 2$. Below we will also use the identity proven in [3]:

$$\chi_{n,t-2m}^{3n,2t}(q) - \chi_{n,t+2m}^{3n,2t}(q) = \chi_{n,m}^{6n,t}(q) - \chi_{n,t-m}^{6n,t}(q) \tag{1.4}$$

where $m < t/2$ and $\langle t, 6 \rangle = \langle n, 2 \rangle = \langle t, n \rangle = 1$.

In some cases characters (1.1) admit the form named ‘fermionic representation’

$$\chi(q) = q^{\text{const}} \sum_{\vec{m}=0}^{\infty} \frac{q^{\vec{m}^t A \vec{m} + \vec{m} \cdot \vec{B}}}{(q)_{m_1} \cdots (q)_{m_r}}. \tag{1.5}$$

* Dedicated to Professor L D Faddeev on his 65th birthday.

Hitherto examples of such representation were obtained for two large classes of characters. For A being related to the inverse Cartan matrix for some Lie algebra they were studied in [4]. In this case certain restrictions are often imposed on the summation over \vec{m} . Examples of (1.5) for $A = 0$ or, at most, being a diagonal matrix (and $(q)_{m_i}$ being replaced with $(q^b)_{m_i}$, $b > 0$) were obtained in [3, 5] as consequences of representation of characters in the ‘factorized form’

$$\chi(q) = q^{\text{const}} \prod_{m=1}^M (\{x_m\}_y^+)^{\gamma_m^+} \prod_{n=1}^N (\{x_n\}_y^-)^{\gamma_n^-} \quad (1.6)$$

where the multiplicities γ_i^\pm are integer. Here and below we use the notation of [3]:

$$\begin{aligned} \{x\}_y^\pm &:= \prod_{k=0}^{\infty} (1 \pm q^{x+ky}) & 0 < x < y \\ \{x_1; \dots; x_n\}_y^\pm &:= \prod_{a=1}^n \{x_a\}_y^\pm. \end{aligned} \quad (1.7)$$

The equivalence of fermionic and factorized forms for characters which admit both types of representation gives rise to nontrivial identities. We will refer to them as to Rogers–Ramanujan-type identities.

The paper is organized as follows. Section 2 contains fermionic sum representations for certain families of characters and linear combinations of characters for $\mathcal{M}(3, t)$. These results are extensions of some previously known examples. Here, several fermionic sum representations for $\mathcal{M}(4, 5)$, $\mathcal{M}(5, 6)$ and $\mathcal{M}(6, 7)$ that seem to be new are also given. In section 3 we observe that all the considered (combinations of) characters also admit the factorized form (1.6) and present the corresponding Rogers–Ramanujan-type identities. In section 4 we apply the saddle point analysis to the fermionic representations given in section 2 and derive Bethe-ansatz-type equations that yield the corresponding effective central charge or (for the differences of characters) the secondary effective charge as a sum of dilogarithms. For the latter case we show how to modify the saddle point analysis if we are dealing with a fermionic sum with alternated signs in summation. We solve the Bethe-ansatz-type equations explicitly and obtain four infinite dilogarithm sum rules corresponding to $\mathcal{M}(3, t)$ and one nontrivial identity for $\mathcal{M}(4, 5)$. In section 5 we employ certain identities for Virasoro characters and expand the list of fermionic representations for $\mathcal{M}(4, 5)$, $\mathcal{M}(5, 6)$ and $\mathcal{M}(6, 7)$. Section 6 contains a brief discussion and conclusion.

2. Fermionic representations for $\mathcal{M}(3, t)$, $\mathcal{M}(4, 5)$, $\mathcal{M}(5, 6)$ and $\mathcal{M}(6, 7)$

It was observed in [4] that the characters $\chi_{1,k}^{3,3k+1}(q)$ and $\chi_{1,k}^{3,3k+2}(q)$ as well as all the four characters of $\mathcal{M}(3, 5)$ admit the fermionic form (1.5) with A_k and \tilde{A}_k given by

$$(A_k)_{ij} = (A_k)_{ji} = \min(i, j) \quad \text{for } 1 \leq i, j \leq k-1 \quad (2.1)$$

$$(A_k)_{kj} = (A_k)_{jk} = \frac{j}{2} + \frac{1-k}{4} \delta_{jk} \quad (2.2)$$

$$(\tilde{A}_k)_{ij} = (A_k)_{ij} - \frac{1}{4} \delta_{ik} \delta_{jk}. \quad (2.3)$$

It turns out that these results can be extended as follows:

$$\chi_{1,n}^{3,3k+1}(q) \pm \chi_{1,3k+1-n}^{3,3k+1}(q) = q^{\eta_{1,n}^{3,3k+1}} \sum_{\vec{m}=0}^{\infty} \frac{(\pm 1)^{m_k} q^{\vec{m}' A_k \vec{m} + \vec{m} \cdot \vec{B}_n^{3k+1}}}{(q)_{m_1} \dots (q)_{m_k}} \quad (2.4)$$

$$\chi_{1,n}^{3,3k+2}(q) \pm \chi_{1,3k+2-n}^{3,3k+2}(q) = q^{\eta_{1,n}^{3,3k+2}} \sum_{\vec{m}=0}^{\infty} \frac{(\pm 1)^{m_k} q^{\vec{m}' \tilde{A}_k \vec{m} + \vec{m} \cdot \vec{B}_n^{3k+2}}}{(q)_{m_1} \dots (q)_{m_k}} \quad (2.5)$$

where $1 \leq n \leq (k+1)$, the matrices A_k and \tilde{A}_k are defined by (2.1)–(2.3) and the corresponding k -component vectors \vec{B}_n^{3k+1} and \vec{B}_n^{3k+2} are such that

$$(\vec{B}_n^{3k+1})_j = \max(j - n + 1, 0) + \frac{n - k - 2}{2} \delta_{jk} \tag{2.6}$$

$$(\vec{B}_n^{3k+2})_j = (\vec{B}_n^{3k+1})_j + \frac{1}{2} \delta_{jk}. \tag{2.7}$$

For example, $\vec{B}_1^{3k+1} = (1, 2, 3, \dots, k - 1, \frac{k-1}{2})$, $\vec{B}_k^{3k+1} = \vec{0}$ and $\vec{B}_{k+1}^{3k+1} = (0, \dots, 0, -\frac{1}{2})$.

It turns out that besides the infinite families (2.4) and (2.5) of fermionic sums for the combinations of characters there exist similar ones for ‘single’ characters:

$$\chi_{1,3t-1}^{3,6t-2}(q) = q^{\eta_{1,3t-1}^{3,6t-2}} \sum_{\vec{m}=0}^{\infty} \frac{q^{\vec{m}' A_{2t-1} \vec{m} + \vec{m} \cdot \vec{C}^{2t-1}}}{(q)_{m_1} \cdots (q)_{m_{2t-1}}} \tag{2.8}$$

$$\chi_{1,3t+1}^{3,6t+2}(q) = q^{\eta_{1,3t+1}^{3,6t+2}} \sum_{\vec{m}=0}^{\infty} \frac{q^{\vec{m}' \tilde{A}_{2t} \vec{m} + \vec{m} \cdot \vec{C}^{2t}}}{(q)_{m_1} \cdots (q)_{m_{2t}}} \tag{2.9}$$

where $t = 1, 2, 3 \dots$, A and \tilde{A} are defined in (2.1)–(2.3), and

$$(\vec{C}^{2t-1})_j = \max(j - t + 1, 0) - \frac{t}{2} \delta_{j,2t-1} \quad (\vec{C}^{2t})_j = \max(j - t + 1, 0) - \frac{t+2}{2} \delta_{j,2t}. \tag{2.10}$$

For example, $\vec{C}^1 = (\frac{1}{2})$, $\vec{C}^2 = (1, \frac{1}{2})$, $\vec{C}^2 = (0, 1, 1)$, $\vec{C}^3 = (0, 1, 2, 1)$.

Although formal derivation of (2.4), (2.5) and (2.8), (2.9) is beyond the scope of this paper, I have verified these identities using Mathematica for $k \leq 7$, $t \leq 5$ expanding them typically up to q^{100} . A rigorous proof can presumably be achieved with the help of the machinery of Bailey pairs (see, e.g., [6]). In contrast to a generic fermionic representation, equations (2.4) and (2.5) do not have restrictions on the summation. However, if we combine them to obtain an expression for a single character, the summation over m_k will be restricted to either odd or even numbers. In particular, equation (2.4) in the case of $k = 1$, $n = 2$ gives two expressions for $\chi_{1,2}^{3,4}(q)$ (as also noted in [4]). Moreover, equation (2.8) yields one more expression for the same character and hence we get

$$q^{-\eta_{1,2}^{3,4}} \chi_{1,2}^{3,4}(q) = \sum_{\substack{m \geq 1 \\ m\text{-odd}}} \frac{q^{\frac{1}{2}m^2 - \frac{1}{2}m}}{(q)_m} = \sum_{\substack{m \geq 0 \\ m\text{-even}}} \frac{q^{\frac{1}{2}m^2 - \frac{1}{2}m}}{(q)_m} = \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m^2 + \frac{1}{2}m}}{(q)_m}. \tag{2.11}$$

While (2.4) and (2.5) extend previously known examples, the following fermionic representations for $\mathcal{M}(4, 5)$, $\mathcal{M}(5, 6)$ and $\mathcal{M}(6, 7)$ to my knowledge have not been discussed in the literature so far.

$\mathcal{M}(4, 5)$. Two characters are representable in the form (1.5) with

$$A = \frac{1}{2} \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.12}$$

namely, for $n = 1, 2$ the following equality holds:

$$\chi_{2,n}^{4,5}(q) = q^{\eta_{1,n}^{4,5}} \sum_{\vec{m}=0}^{\infty} \frac{q^{2m_1^2 + \frac{1}{2}m_2^2 + m_1 m_2 + (4-2n)m_1 + \frac{1}{2}m_2}}{(q)_{m_1} (q)_{m_2}}. \tag{2.13}$$

$\mathcal{M}(5, 6)$. Certain linear combinations of characters admit the form (1.5) with

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (2.14)$$

Namely, for $n = 1, 2$, we have

$$\chi_{n,2}^{5,6}(q) \pm \chi_{n,4}^{5,6}(q) = q^{\eta_{n,2}^{5,6}} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{(\pm 1)^{m_2} q^{\frac{1}{2}(m_1^2+m_2^2)+m_1 m_2+\frac{1}{2}m_1+(2-n)m_2}}{(q)_{m_1}(q)_{m_2}} \quad (2.15)$$

$$\chi_{n,1}^{5,6}(q) - \chi_{n,5}^{5,6}(q) = q^{\eta_{n,1}^{5,6}} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{(-1)^{m_2} q^{\frac{1}{2}(m_1^2+m_2^2)+m_1 m_2+(n-1)(m_1+m_2)}}{(q)_{m_1}(q)_{m_2}}. \quad (2.16)$$

$\mathcal{M}(5, 6)$. Another fermionic representation for the characters of the $\mathcal{M}(5, 6)$ model can be obtained if we notice that equations (1.3) and (1.4) allow us to relate these characters to those of the $\mathcal{M}(3, 10)$ model: $\chi_{n,2}^{5,6}(q) \pm \chi_{n,4}^{5,6}(q) = \chi_{1,2n}^{3,10}(q) \pm \chi_{1,10-2n}^{3,10}(q)$ and $\chi_{n,1}^{5,6}(q) - \chi_{n,5}^{5,6}(q) = \chi_{1,5-2n}^{3,10}(q) - \chi_{1,5+2n}^{3,10}(q)$, $n = 1, 2$. Combining these relations with formulae (2.4) for $k = 3$, we find

$$\chi_{n,2}^{5,6}(q) \pm \chi_{n,4}^{5,6}(q) = q^{\eta_{n,2}^{5,6}} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{(\pm 1)^{m_3} q^{\vec{m}^t A \vec{m} + (2-n)m_2 + (\frac{3}{2}-n)m_3}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}} \quad (2.17)$$

$$\chi_{n,1}^{5,6}(q) - \chi_{n,5}^{5,6}(q) = q^{\eta_{n,1}^{5,6}} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{(-1)^{m_3} q^{\vec{m}^t A \vec{m} + (n-1)(m_1+2m_2+m_3)}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}} \quad (2.18)$$

where $n = 1, 2$ and

$$A = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & 1 \end{pmatrix}. \quad (2.19)$$

Furthermore, equation (1.3) also implies that $\chi_{n,3}^{5,6}(q) = \chi_{1,3n}^{2,15}(q)$, $n = 1, 2$. For the $\mathcal{M}(2, 2k+1)$ model a fermionic representation is well known [7–9]:

$$\chi_{1,n}^{2,2k+1}(q) = q^{\eta_{1,n}^{2,2k+1}} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{q^{\vec{m}^t \bar{A}_k \vec{m} + \vec{m} \cdot \vec{F}_n^k}}{(q)_{m_1} \cdots (q)_{m_{k-1}}} \quad (2.20)$$

where $1 \leq n \leq k$, $(\vec{F}_n^k)_j = \max(j - n + 1, 0)$, and \bar{A}_k coincides with the $(k-1) \times (k-1)$ minor (2.1) which is the inverse Cartan matrix of the tadpole graph with $(k-1)$ nodes. This gives us yet another fermionic representation for $\mathcal{M}(5, 6)$ with A as the 6×6 matrix \bar{A}_7 .

$\mathcal{M}(6, 7)$. Employing again equations (1.3) and (1.4), we observe that $\chi_{2,n}^{6,7}(q) \pm \chi_{4,n}^{6,7}(q) = \chi_{1,2n}^{3,14}(q) \pm \chi_{1,14-2n}^{3,14}(q)$ and $\chi_{1,n}^{6,7}(q) - \chi_{5,n}^{6,7}(q) = \chi_{1,7-2n}^{3,14}(q) - \chi_{1,7+2n}^{3,14}(q)$, where $n = 1, 2, 3$. Combining these relations with formulae (2.5) for $k = 4$, we find

$$\chi_{2,n}^{6,7}(q) \pm \chi_{4,n}^{6,7}(q) = q^{\eta_{2,n}^{6,7}} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{(\pm 1)^{m_4} q^{\vec{m}^t A \vec{m} + (2-n)(m_2+2m_3) + (\frac{5}{2}-n)m_4}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}(q)_{m_4}} \quad n = 1, 2 \quad (2.21)$$

$$\chi_{1,n}^{6,7}(q) - \chi_{5,n}^{6,7}(q) = q^{\eta_{1,n}^{6,7}} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{(-1)^{m_4} q^{\vec{m}^t A \vec{m} + \vec{m} \cdot \vec{D}^n}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}(q)_{m_4}} \quad n = 1, 2, 3 \quad (2.22)$$

where $\vec{D}^1 = (0, 0, 0, 0)$, $\vec{D}^2 = (0, 0, 1, 1)$, $\vec{D}^3 = (1, 2, 3, 2)$ and

$$A = \begin{pmatrix} 1 & 1 & 1 & \frac{1}{2} \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 3 & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & 1 \end{pmatrix}. \tag{2.23}$$

Furthermore, due to equation (1.3) we can identify $\chi_{3,n}^{6,7}(q) = \chi_{1,3n}^{2,21}(q)$, $n = 1, 2, 3$. Therefore, these three characters of $\mathcal{M}(6, 7)$ admit a fermionic form of the type (2.20) with A as the 9×9 matrix \bar{A}_{10} .

3. Rogers–Ramanujan-type identities

In the previous section we considered some characters and combinations of characters which possess fermionic representations. It turns out that all of them have another common feature—they are factorizable, that is they also admit the form (1.6). For the characters of the $\mathcal{M}(2, 2k+1)$ models this is the well known representation

$$\chi_{1,n}^{2,2k+1}(q) = q^{n_{1,n}^{2,2k+1}} \prod_{\substack{j=1 \\ j \neq n}}^k \frac{1}{\{j; 2k+1-j\}_{2k+1}^-} \tag{3.1}$$

where we use the notation of (1.7). Equality of the rhs of equations (3.1) and (2.20) yields a family of identities known as the Andrews–Gordon identities [7]. For $k = 2$, these are the famous Rogers–Ramanujan identities [10]

$$\sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q)_m} = \frac{1}{\{2; 3\}_5^-} \quad \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m} = \frac{1}{\{1; 4\}_5^-}. \tag{3.2}$$

Actually, (3.1) is only a particular case of a more general formula

$$\chi_{n,m}^{2n,t}(q) = \frac{q^{n_{n,m}^{2n,t}}}{\{1\}_1^-} \{nm; nt - nm; nt\}_{nt}^- \tag{3.3}$$

which together with

$$\chi_{n,m}^{3n,t}(q) = \frac{q^{n_{n,m}^{3n,t}}}{\{1\}_1^-} \{nm; 2nt - nm; 2nt\}_{2nt}^- \{2nt - 2nm; 2nt + 2nm\}_{4nt}^- \tag{3.4}$$

exhausts the possibility for single characters to be factorizable on the basis of the $A_1^{(1)}$ and $A_2^{(2)}$ Macdonald identities [3, 11, 12]. Furthermore, it was shown in [3] that in certain cases the following combinations:

$$\chi_{n,m}^{s,t}(q) \pm \chi_{n,t-m}^{s,t}(q) \tag{3.5}$$

also are factorizable on the basis of the same Macdonald identities. The explicit formulae found in [3] read

$$\chi_{n,m}^{3n,t}(q) \pm \chi_{2n,m}^{3n,t}(q) = \frac{q^{n_{n,m}^{3n,t}}}{\{1\}_1^-} \{nm; nt - nm\}_{nt}^- \left\{ \frac{nt}{2} \right\}_{\frac{nt}{2}}^- \left\{ \frac{nt - 2nm}{4}; \frac{nt + 2nm}{4} \right\}_{\frac{nt}{2}}^\pm \tag{3.6}$$

$$\chi_{n,m}^{4n,t}(q) \pm \chi_{3n,m}^{4n,t}(q) = \frac{q^{n_{n,m}^{4n,t}}}{\{1\}_1^-} \{nm; nt - nm; nt\}_{nt}^- \left\{ \frac{nt}{2} - nm; \frac{nt}{2} + nm; \frac{nt}{2} \right\}_{nt}^\pm \tag{3.7}$$

$$\chi_{n,m}^{6n,t}(q) - \chi_{5n,m}^{6n,t}(q) = \frac{q^{n_{n,m}^{6n,t}}}{\{1\}_1^-} \{nm; nt - nm; nt\}_{nt}^- \{nt - 2nm; nt + 2nm\}_{2nt}^- \tag{3.8}$$

Combining the fermionic representations given in the previous section with these factorized representations, we obtain various identities of the ‘sum–product’ type. They can be regarded as generalizations of the Rogers–Ramanujan identities and it seems that many of them (especially those with a multivariable summation) have not previously appeared in the literature.

$\mathcal{M}(3, t)$. According to (3.6) the fermionic sums on the rhs of (2.4), (2.5) are equal, respectively, to

$$\frac{q^{n_{1,n}^{3,3k+1}}}{\{1\}_1^-} \{n; 3k + 1 - n\}_{3k+1}^- \left\{ \frac{3k + 1}{2} \right\}_{\frac{3k+1}{2}}^- \left\{ \frac{3k + 1 - 2n}{4}; \frac{3k + 1 + 2n}{4} \right\}_{\frac{3k+1}{2}}^\pm \tag{3.9}$$

$$\frac{q^{n_{1,n}^{3,3k+2}}}{\{1\}_1^-} \{n; 3k + 2 - n\}_{3k+2}^- \left\{ \frac{3k + 2}{2} \right\}_{\frac{3k+2}{2}}^- \left\{ \frac{3k + 2 - 2n}{4}; \frac{3k + 2 + 2n}{4} \right\}_{\frac{3k+2}{2}}^\pm . \tag{3.10}$$

For instance, for $k = 1, 2$ we get the following identities:

$$\sum_{m=0}^\infty \frac{(\pm 1)^m q^{\frac{1}{2}m^2}}{(q)_m} = \left\{ \frac{1}{2} \right\}_1^\pm \sum_{m=0}^\infty \frac{(\pm 1)^m q^{\frac{1}{2}m^2 - \frac{1}{2}m}}{(q)_m} = (1 \pm 1) \{1\}_1^+ \tag{3.11}$$

$$\sum_{m=0}^\infty \frac{(\pm 1)^m q^{\frac{1}{4}m^2 + (1 - \frac{n}{2})m}}{(q)_m} = \frac{\left\{ \frac{5-2n}{4}; \frac{5+2n}{4} \right\}_{\frac{n}{2}}^\pm}{\left\{ \frac{5}{2} \right\}_{\frac{n}{2}}^+ \{3 - n; 2 + n\}_5^-} \quad n = 1, 2 \tag{3.12}$$

$$\sum_{\vec{m}=0}^\infty \frac{(\pm 1)^{m_2} q^{m_1^2 + \frac{3}{4}m_2 + m_1m_2 + \delta_{1n}m_1 + (1 - \frac{n}{2})m_2}}{(q)_{m_1} (q)_{m_2}} = \frac{\left\{ \frac{7-2n}{4}; \frac{7+2n}{4} \right\}_{\frac{n}{2}}^\pm}{\left\{ \frac{7}{2} \right\}_{\frac{n}{2}}^+ \prod_{j \neq n}^3 \{j; 7 - j\}_7^-} \quad 1 \leq n \leq 3 \tag{3.13}$$

$$\sum_{\vec{m}=0}^\infty \frac{(\pm 1)^{m_2} q^{m_1^2 + \frac{1}{2}m_2 + m_1m_2 + \delta_{1n}m_1 + (\frac{3}{2} - \frac{n}{2})m_2}}{(q)_{m_1} (q)_{m_2}} = \frac{\left\{ 2 - \frac{n}{2}; 2 + \frac{n}{2} \right\}_4^\pm}{\prod_{j=1}^3 \{j; 8 - j\}_8^-} \quad 1 \leq n \leq 3. \tag{3.14}$$

Here we simplified the product sides exploiting the Euler identity $\{x\}_x^+ \{x\}_{2x}^- = 1$ and other transformations (see [3]). Equations (3.11) are well known (see, e.g., [13]) and equations (3.12) were presented in [3]. It should be remarked that in some cases the combinations on the lhs of (2.4), (2.5) belong both to (3.6) and (3.7). In this case the product side acquires a more compact form [3]:

$$\chi_{m,n}^{3m,4n}(q) \pm \chi_{m,3n}^{3m,4n}(q) = \frac{q^{\frac{nm-2}{48}}}{\{1\}_1^-} \{nm\}_{nm}^- \left\{ \frac{nm}{2} \right\}_{nm}^\pm . \tag{3.15}$$

Therefore, we obtain

$$\sum_{\vec{m}=0}^\infty \frac{(\pm 1)^{m_{4t+1}} q^{\vec{m}^t A_{4t+1} \vec{m} + \vec{m} \cdot \vec{B}_{3t+1}^{12t+4}}}{(q)_{m_1} \dots (q)_{m_{4t+1}}} = \frac{\{3t + 1\}_{3t+1}^- \left\{ \frac{3t+1}{2} \right\}_{3t+1}^\pm}{\{1\}_1^-} \tag{3.16}$$

$$\sum_{\vec{m}=0}^\infty \frac{(\pm 1)^{m_{4t+2}} q^{\vec{m}^t \tilde{A}_{4t+2} \vec{m} + \vec{m} \cdot \tilde{B}_{3t+2}^{12t+8}}}{(q)_{m_1} \dots (q)_{m_{4t+2}}} = \frac{\{3t + 2\}_{3t+2}^- \left\{ \frac{3t+2}{2} \right\}_{3t+2}^\pm}{\{1\}_1^-} \tag{3.17}$$

where $t = 0, 1, 2, \dots$, and A, \tilde{A} and \tilde{B} are defined in (2.1)–(2.3) and (2.6), (2.7). For instance, (3.17) yields for $t = 0$,

$$\sum_{\vec{m}=0}^\infty \frac{q^{m_1^2 + \frac{1}{2}m_2^2 + m_1m_2 + \frac{1}{2}m_2}}{(q)_{m_1} (q)_{m_2}} = \{1\}_1^+ \{1\}_2^+ \sum_{\vec{m}=0}^\infty \frac{(-1)^{m_2} q^{m_1^2 + \frac{1}{2}m_2^2 + m_1m_2 + \frac{1}{2}m_2}}{(q)_{m_1} (q)_{m_2}} = 1. \tag{3.18}$$

The last equality is due to the fact that $\chi_{1,2}^{3,8}(q) - \chi_{1,6}^{3,8}(q) = 1$ (see [3, 12]).

For the fermionic sums (2.8), (2.9) the product side also simplifies since these characters belong both to (3.3) and (3.4). Namely, we obtain for $t = 1, 2, 3 \dots$

$$\sum_{\vec{m}=0}^{\infty} \frac{q^{\vec{m}' A_{2t-1} \vec{m} + \vec{m} \cdot \vec{C}^{2t-1}}}{(q)_{m_1} \dots (q)_{m_{2t-1}}} = \frac{\{3t-1\}_{3t-1}^-}{\{1\}_1^-} \quad \sum_{\vec{m}=0}^{\infty} \frac{q^{\vec{m}' \tilde{A}_2 \vec{m} + \vec{m} \cdot \vec{C}^{2t}}}{(q)_{m_1} \dots (q)_{m_{2t}}} = \frac{\{3t+1\}_{3t+1}^-}{\{1\}_1^-} \quad (3.19)$$

where A, \tilde{A} and \vec{C} are defined in (2.1)–(2.3) and (2.10). For instance, for $t = 1$, we get

$$\sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m^2 + \frac{1}{2}m}}{(q)_m} = \{1\}_1^+ \quad \sum_{m=0}^{\infty} \frac{q^{m^2 + \frac{1}{2}m^2 + m_1 m_2 + m_1 + \frac{1}{2}m_2}}{(q)_{m_1} (q)_{m_2}} = \{1\}_1^+ \{2\}_2^+. \quad (3.20)$$

To complete the discussion of the $\mathcal{M}(3, t)$ case, we recall that since each of equations (2.4), (2.5) comprises two equalities, we can express the involved characters as sums without alternation of the sign, but with the summation over m_k restricted to odd or even numbers. Then, according to (3.4), each character obtained in this way also admits the product representation. Thus, we get for $t = 3k + 1, 1 \leq n \leq k + 1$

$$\sum_{\substack{\vec{m}=0 \\ m_k \text{ - even}}}^{\infty} \frac{q^{\vec{m}' A_k \vec{m} + \vec{m} \cdot \vec{B}'_n}}{(q)_{m_1} \dots (q)_{m_k}} = \frac{1}{\{1\}_1} \{n; 2t - n; 2t\}_{2t}^- \{2t - 2n; 2t + 2n\}_{4t}^- \quad (3.21)$$

$$\sum_{\substack{\vec{m}=0 \\ m_k \text{ - odd}}}^{\infty} \frac{q^{\vec{m}' A_k \vec{m} + \vec{m} \cdot \vec{B}'_n}}{(q)_{m_1} \dots (q)_{m_k}} = \frac{1}{\{1\}_1} \{t - n; t + n; 2t\}_{2t}^- \{2n; 4t - 2n\}_{4t}^- \quad (3.22)$$

and analogous formulae for $t = 3k + 2$ if A_k is replaced by \tilde{A}_k .

$\mathcal{M}(4, 5)$. From (2.13) and (3.3) we obtain for $n = 1, 2$

$$\sum_{\vec{m}=0}^{\infty} \frac{q^{2m_1^2 + \frac{1}{2}m_2^2 + m_1 m_2 + (4-2n)m_1 + \frac{1}{2}m_2}}{(q)_{m_1} (q)_{m_2}} = \frac{\{5\}_5^+}{\{n; 5 - n\}_5^- \{5 - 2n; 5 + 2n\}_{10}^-}. \quad (3.23)$$

$\mathcal{M}(5, 6)$. Combining (2.15)–(2.18) with (3.6) and (3.8), we obtain for $n = 1, 2$

$$\begin{aligned} \sum_{\vec{m}=0}^{\infty} \frac{(\pm 1)^{m_2} q^{\frac{1}{2}(m_1^2 + m_2^2) + m_1 m_2 + \frac{1}{2}m_1 + (2-n)m_2}}{(q)_{m_1} (q)_{m_2}} &= \frac{1}{\{n; 5 - n\}_5^- \{\frac{5}{2} - n; \frac{5}{2} + n\}_5^{\mp}} \\ &= \sum_{\vec{m}=0}^{\infty} \frac{(\pm 1)^{m_3} q^{m_1^2 + 2m_2^2 + m_3^2 + 2m_1 m_2 + m_1 m_3 + 2m_2 m_3 + (2-n)m_2 + (\frac{3}{2}-n)m_3}}{(q)_{m_1} (q)_{m_2} (q)_{m_3}} \end{aligned} \quad (3.24)$$

$$\begin{aligned} \sum_{\vec{m}=0}^{\infty} \frac{(-1)^{m_2} q^{\frac{1}{2}(m_1^2 + m_2^2) + m_1 m_2 + (n-1)(m_1 + m_2)}}{(q)_{m_1} (q)_{m_2}} &= \frac{1}{\{2n; 10 - 2n\}_{10}^-} \\ &= \sum_{\vec{m}=0}^{\infty} \frac{(-1)^{m_3} q^{m_1^2 + 2m_2^2 + m_3^2 + 2m_1 m_2 + m_1 m_3 + 2m_2 m_3 + (n-1)(m_1 + 2m_2 + m_3)}}{(q)_{m_1} (q)_{m_2} (q)_{m_3}}. \end{aligned} \quad (3.25)$$

$\mathcal{M}(6, 7)$. The fermionic sums on the rhs of (2.21) and (2.22) are equal, respectively, to

$$\frac{q^{\eta_{2,n}^{6,7}}}{\{1; 4 - n; 3 + n; 6\}_7^- \{\frac{7}{2} - n; \frac{7}{2} + n\}_7^{\mp}} \quad \frac{q^{\eta_{1,n}^{6,7}}}{\{d_n; 7 - d_n\}_7^- \{2n; 14 - 2n\}_{14}^-} \quad (3.26)$$

where $d_1 = 3, d_2 = 1, d_3 = 2$.

4. Dilogarithm identities

If a q -series $\chi(q)$ (not necessarily identified in terms of characters) admits both fermionic representation (1.5) and product representation (1.6), it implies not only the existence of a Rogers–Ramanujan-type identity but also leads to a certain identity involving the dilogarithm function, $L(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2} \ln x \ln(1-x)$. Indeed, the product side allows us to find easily the number c_{eff} that governs the asymptotics of $\chi(q)$ in the $q \rightarrow 1$ limit (see, e.g., [3, 5]). On the other hand, the same number can be obtained from the fermionic sum by saddle point analysis (see, e.g., [4, 9]). Equivalence of the two expressions for c_{eff} is typically a nontrivial identity. Of course, if it is known *a priori* that $\chi(q)$ is a character, then its fermionic form alone leads to a dilogarithm identity since c_{eff} (effective central charge) is fixed by the properties of $\chi(q)$ with respect to the modular transformations. Namely, let $q = e^{2\pi i\tau}$ and $\hat{q} = e^{-2\pi i/\tau}$, then for the minimal Virasoro model $\mathcal{M}(s, t)$ we have $\chi_{n,m}^{s,t}(q) \sim \hat{q}^{-\frac{c_{\text{eff}}(s,t)}{24}}$ as $q \rightarrow 1$, where

$$c_{\text{eff}}(s, t) = c(s, t) - 24h' = 1 - \frac{6}{st}. \tag{4.1}$$

Here h' denotes the lowest conformal weight in the model. Furthermore, as it was shown in [3], a difference of characters of the type (3.5) for all minimal models but $\mathcal{M}(2, t)$ has the asymptotics $\hat{q}^{-\frac{\tilde{c}(s,t)}{24}}$ when $q \rightarrow 1$. Here \tilde{c} (secondary effective central charge) is given by

$$\tilde{c}(s, t) = c(s, t) - 24h'' = 1 - \frac{24}{st} \tag{4.2}$$

where h'' stands for the second lowest conformal weight in the model.

For our purposes we need to consider a slightly generalized version of (1.5):

$$\chi(q) = q^{\text{const}} \sum_{\vec{m}} \frac{q^{\vec{m}^t A \vec{m} + \vec{m} \cdot \vec{B}}}{(q^{b_1})_{m_1} \dots (q^{b_r})_{m_r}} \tag{4.3}$$

where b_i are some positive numbers. Modifying properly the standard saddle point analysis of a fermionic sum (see [4, 9] for the case $b_i = 1$ and [5] for $b_i = b \neq 1$), we find that (4.3) has the asymptotics $\hat{q}^{-\frac{c_{\text{eff}}}{24}}$ as $q \rightarrow 1$ with

$$c_{\text{eff}} = \frac{6}{\pi^2} \sum_{i=1}^r \frac{1}{b_i} L(x_i). \tag{4.4}$$

Here the set of numbers $0 < x_i < 1$ satisfies the following equations:

$$x_i = \prod_{j=1}^r (1 - x_j)^{\frac{1}{b_j} (A_{ij} + A_{ji})} \quad i = 1, \dots, r. \tag{4.5}$$

Let us define $(\hat{A})_{ij} = \frac{1}{2b_j} (A_{ij} + A_{ji})$. If matrix \hat{A} is invertible, it is convenient to introduce $I = 2 - (\hat{A})^{-1}$ (generalized incidence matrix) and make the substitution $x_i = 1/\mu_i^2$. Then (4.4) and (4.5) turn into

$$c_{\text{eff}} = \frac{6}{\pi^2} \sum_{i=1}^r \frac{1}{b_i} L\left(\frac{1}{\mu_i^2}\right) \quad \mu_i^2 = 1 + \prod_{j=1}^r (\mu_j)^{I_{ij}}. \tag{4.6}$$

As we have seen above, certain differences of characters of the type (3.5) admit the fermionic form with alternated summation over the last variable,

$$\chi(q) = q^{\text{const}} \sum_{\vec{m}} \frac{(-1)^{m_r} q^{\vec{m}^t A \vec{m} + \vec{m} \cdot \vec{B}}}{(q)_{m_1} \dots (q)_{m_r}}. \tag{4.7}$$

Let us find equations describing the $q \rightarrow 1$ limit of such series. To this end we notice that

$$\frac{1}{(q)_m} = (-1)^m q^{-\frac{1}{2}m(m+1)} \frac{1}{(q^{-1})_m}. \tag{4.8}$$

Therefore, we can rewrite (4.7) as follows:

$$\chi(q) = q^{\text{const}} \sum_{\vec{m}} \frac{q^{\vec{m}' \cdot A' \vec{m} + \vec{m} \cdot \vec{B}'}}{(q)_{m_1} \cdots (q)_{m_{r-1}} (q^{-1})_{m_r}} \tag{4.9}$$

where

$$(A')_{ij} = A_{ij} - \frac{1}{2} \delta_{ir} \delta_{jr} \quad (\vec{B}')_j = (\vec{B})_j - \frac{1}{2} \delta_{jr}. \tag{4.10}$$

Equation (4.9) is a particular case of (4.3) with $b_1 = \cdots = b_{r-1} = -b_r = 1$ and thus we can apply equations (4.4), (4.5). We conclude that (4.7) has the asymptotics $\hat{q}^{-\frac{\tilde{c}}{24}}$ as $q \rightarrow 1$ with

$$\tilde{c} = \frac{6}{\pi^2} \left(\sum_{i=1}^{r-1} L(y_i) - L(y_r) \right) \tag{4.11}$$

where the set of numbers $0 < y_i < 1$ satisfies the following equations:

$$y_i = \prod_{j=1}^r (1 - y_j)^{(-1)^{\delta_{jr}} (A'_{ij} + A'_{ji})} \quad i = 1, \dots, r. \tag{4.12}$$

It is again convenient to introduce $I' = 2 - (\hat{A}')^{-1}$, where $(\hat{A}')_{ij} = \frac{1}{2}(-1)^{\delta_{jr}} (A'_{ij} + A'_{ji})$. Then, making the substitution $y_i = 1/v_i^2$, we transform (4.11), (4.12) to

$$\tilde{c} = \frac{6}{\pi^2} \left(\sum_{i=1}^{r-1} L\left(\frac{1}{v_i^2}\right) - L\left(\frac{1}{v_r^2}\right) \right) \quad v_i^2 = 1 + \prod_{j=1}^r (v_j)^{I'_{ij}}. \tag{4.13}$$

Let us remark that performing the following change of variables in (4.11), (4.12): $z_i = y_i$, $i < r$ and $z_r = \frac{y_r}{y_r - 1}$, we can transform these equations to the form almost coinciding with (4.4), (4.5) (for $b_i = 1$) and involving the initial matrix A :

$$\tilde{c} = \frac{6}{\pi^2} \sum_{i=1}^r L(z_i) \quad (-1)^{\delta_{ir}} z_i = \prod_{j=1}^r (1 - z_j)^{A_{ij} + A_{ji}} \quad i = 1, \dots, r. \tag{4.14}$$

In contrast to (4.5), now $z_r < 0$. Deriving (4.14) we used the definition [14, 15]: $L(x) = L(\frac{1}{1-x}) - L(1)$ for $x < 0$, and the property $L(x) = -L(\frac{x}{x-1})$ for $x < 1$.

Generalization of (4.11)–(4.14) for the case of a fermionic sum involving alternated summation over several variables is obvious. Now let us list dilogarithm identities that follow from the formulae (4.6), (4.13) and (4.1), (4.2) for the (combinations of) characters considered in the previous sections.

$\mathcal{M}(3, 3k+1)$ and $\mathcal{M}(3, 3k+2)$. The explicit expressions (2.1)–(2.3) for the matrices A_k and \tilde{A}_k allow us to compute the corresponding matrices I_k and \tilde{I}_k

$$(I_k)_{ij} = \delta_{i,j+1} + \delta_{i+1,j} + \frac{1}{2} \delta_{i,k-1} \delta_{j,k-1} \tag{4.15}$$

$$(\tilde{I}_k)_{ij} = \delta_{i,j+1} + \delta_{i+1,j} + \delta_{i,k} \delta_{j,k-1} + \delta_{i,k-1} \delta_{j,k} - 2\delta_{i,k} \delta_{j,k}. \tag{4.16}$$

These generalized incidence matrices differ from those of the Lie algebra A_k only by a few entries in the lower-right corner. This hints of a possibility to solve the corresponding sets of equations (4.6) in a uniform manner, similar to that known for the $\mathcal{M}(2, t)$ models [9, 16].

Indeed, we find the following solutions of (4.6) (they can be verified by a straightforward substitution):

$$\mu_i = \frac{\sin \frac{(i+1)\pi}{3k+1}}{\sin \frac{\pi}{3k+1}} \quad 1 \leq i \leq k-1 \quad \mu_k^2 = 1 + \frac{\sin \frac{k\pi}{3k+1}}{\sin \frac{\pi}{3k+1}} \quad (4.17)$$

for I_k given by (4.15) (that is in the case of $\mathcal{M}(3, 3k+1)$ model) and

$$\mu_i = \frac{\sin \frac{(i+1)\pi}{3k+2}}{\sin \frac{\pi}{3k+2}} \quad 1 \leq i \leq k-1 \quad \mu_k^2 = \frac{\sin \frac{(k+1)\pi}{3k+2}}{\sin \frac{\pi}{3k+2}} \quad (4.18)$$

for \tilde{I}_k given by (4.16) ($\mathcal{M}(3, 3k+2)$ model). Combining these results with (4.1), we derive the following identities:

$$\sum_{i=1}^{k-1} L \left(\frac{\sin^2 \frac{\pi}{3k+1}}{\sin^2 \frac{(i+1)\pi}{3k+1}} \right) + L \left(\frac{\sin \frac{\pi}{3k+1}}{\sin \frac{\pi}{3k+1} + \sin \frac{k\pi}{3k+1}} \right) = \frac{\pi^2}{6} \frac{3k-1}{3k+1} \quad (4.19)$$

$$\sum_{i=1}^{k-1} L \left(\frac{\sin^2 \frac{\pi}{3k+2}}{\sin^2 \frac{(i+1)\pi}{3k+2}} \right) + L \left(\frac{\sin \frac{\pi}{3k+2}}{\sin \frac{(k+1)\pi}{3k+2}} \right) = \frac{\pi^2}{6} \frac{3k}{3k+2}. \quad (4.20)$$

Let us remark that although these identities were derived here exploiting the modular properties of characters and the saddle point analysis, they resemble the ‘general A_1 -type’ dilogarithm identities [15] and probably can be proved in more direct way based on the functional relations for the dilogarithm. For instance, equation (4.20) for $k = 2$ yields the equality $(L(1 - \frac{1}{\sqrt{2}}) + L(\sqrt{2} - 1)) = \frac{\pi^2}{8}$. It can be proved with the help of the Abel duplication formula [14, 15]. It should be mentioned that equation (4.19) for k odd was encountered in [17] in the context of the thermodynamic Bethe ansatz.

Next we consider the differences of characters in (2.4), (2.5). First, we compute the matrices I'_k and \tilde{I}'_k . It turns out that for $\mathcal{M}(3, 3k+1)$ $\det(\hat{A}'_k) = 0$, i.e. I'_k does not exist and we have to solve the equations (4.12). For $\mathcal{M}(3, 3k+2)$ the matrix \tilde{I}'_k exists and is given by

$$(\tilde{I}'_k)_{ij} = \delta_{i,j+1} + \delta_{i+1,j} + \delta_{i,k} \delta_{j,k-1} - 3\delta_{i,k-1} \delta_{j,k} + 2\delta_{i,k-1} \delta_{j,k-1} - 2\delta_{i,k} \delta_{j,k}. \quad (4.21)$$

We obtain the following solution of (4.12) for $\mathcal{M}(3, 3k+1)$ (written in terms of $v_i = 1/\sqrt{y_i}$ for the sake of uniformness):

$$v_i = \frac{\sin \frac{(2i+2)\pi}{3k+1}}{\sin \frac{2\pi}{3k+1}} \quad 1 \leq i \leq k-1 \quad v_k^2 = \frac{\sin \frac{2k\pi}{3k+1}}{\sin \frac{2\pi}{3k+1}} \quad (4.22)$$

and of (4.13) for $\mathcal{M}(3, 3k+2)$:

$$v_i = \frac{\sin \frac{(2i+2)\pi}{3k+2}}{\sin \frac{2\pi}{3k+2}} \quad 1 \leq i \leq k-1 \quad v_k^2 = 1 + \frac{\sin \frac{(2k+2)\pi}{3k+2}}{\sin \frac{2\pi}{3k+2}}. \quad (4.23)$$

Combining these results with (4.2), we obtain the following dilogarithm identities:

$$\sum_{i=1}^{k-1} L \left(\frac{\sin^2 \frac{2\pi}{3k+1}}{\sin^2 \frac{(2i+2)\pi}{3k+1}} \right) - L \left(\frac{\sin \frac{2\pi}{3k+1}}{\sin \frac{2k\pi}{3k+1}} \right) = \frac{\pi^2}{6} \frac{3k-7}{3k+1} \quad (4.24)$$

$$\sum_{i=1}^{k-1} L \left(\frac{\sin^2 \frac{2\pi}{3k+2}}{\sin^2 \frac{(2i+2)\pi}{3k+2}} \right) - L \left(\frac{\sin \frac{2\pi}{3k+2}}{\sin \frac{2\pi}{3k+2} + \sin \frac{(2k+2)\pi}{3k+2}} \right) = \frac{\pi^2}{6} \frac{3k-6}{3k+2}. \quad (4.25)$$

$\mathcal{M}(4, 5)$. Equation (4.5) for A given by (2.12) can be reduced to a bi-quadratic equation. Solving it we obtain $x_1 = 1 - \sqrt{\rho}$, $x_2 = 1 - \frac{1}{1+\sqrt{\rho}}$, where $\rho := \frac{\sqrt{5}-1}{2}$. According to (4.1) and (4.4) this leads to the identity

$$L\left(1 - \sqrt{\frac{\sqrt{5}-1}{2}}\right) + L\left(1 - \frac{\sqrt{2}}{\sqrt{2} + \sqrt{\sqrt{5}-1}}\right) = \frac{7}{10} \frac{\pi^2}{6}. \tag{4.26}$$

Although the summation in the fermionic representation (2.13) is not alternated, we can nevertheless introduce matrix A' according to (4.10) and solve equations (4.11), (4.12). The solution is $y_1 = 1 - \rho$, $y_2 = \rho$ and thus we get $\tilde{c} = \frac{6}{\pi^2}(L(1 - \rho) - L(\rho)) = -\frac{1}{5}$. This identity is rather trivial mathematically (since it is well known that $L(\rho) = \frac{\pi^2}{10}$ and $L(1 - \rho) = \frac{\pi^2}{15}$) but it is remarkable that the value of \tilde{c} is in agreement with (4.2).

$\mathcal{M}(5, 6)$. It is easy to see that for this model any matrix of the form

$$A = \begin{pmatrix} 1 - \alpha & \alpha \\ \alpha & 1 - \alpha \end{pmatrix} \quad \alpha \leq \frac{1}{2} \tag{4.27}$$

yields $x_1 = x_2 = 1 - \rho$ and gives the correct central charge: $c_{\text{eff}} = \frac{6}{\pi^2} 2L(1 - \rho) = \frac{4}{5}$. Besides our example (2.14) corresponding to $\alpha = \frac{1}{2}$, the case of $\alpha = \frac{1}{3}$ is known [4]. However, matrix A' constructed from (4.27) according to (4.10) leads to the correct value of \tilde{c} only for $\alpha = \frac{1}{2}$. Namely, in this case we get $y_1 = \rho$, $y_2 = 1 - \rho$ and $\tilde{c} = \frac{6}{\pi^2}(L(\rho) - L(1 - \rho)) = \frac{1}{5}$. It would be interesting to see if there are fermionic representations for characters of $\mathcal{M}(5, 6)$ corresponding to other values of α . Below we will show that $\alpha = 0$ appears not in $\mathcal{M}(5, 6)$ but in the closely related $\mathcal{M}(3, 10)$ model.

5. Further fermionic representations and identities

So far, we have considered only ‘irreducible’ fermionic representations of characters, i.e. those that are not decomposable into a product of other fermionic sums. However, there are many ways to construct ‘reducible’ fermionic representations. One of them was discussed in [3, 5] and was based on the fact that any factorizable character can be brought to the form (4.3) (typically with $b_i \neq 1$) with the help of the following formulae:

$$\{x\}_y^\pm = \sum_{m=0}^\infty \frac{(\pm 1)^m q^{\frac{y}{2}(m^2 - m) + mx}}{(q^y)_m} \quad \frac{1}{\{x\}_y^\pm} = \sum_{m=0}^\infty \frac{(\mp 1)^m q^{mx}}{(q^y)_m}. \tag{5.1}$$

Another possibility is to use relations expressing a character as a product of other characters for which fermionic representation is known. To derive and prove such relations it is often convenient to exploit the factorized forms of characters. For instance, with the help of (3.3), (3.4) and (3.6) it is straightforward to check the following identities (see [3] for a similar derivation):

$$\chi_{n,m}^{2n,3m}(q) \chi_{n,5m}^{3n,10m}(q) = \chi_{n,m}^{2n,5m}(q) \chi_{n,2m}^{2n,5m}(q) \tag{5.2}$$

$$\chi_{n,m}^{2n,3m}(q) (\chi_{n,m(2k-1)}^{3n,10m}(q) + \chi_{n,m(11-2k)}^{3n,10m}(q)) = \chi_{n,mk}^{2n,5m}(q) \chi_{n,mk}^{2n,5m}(q) \quad k = 1, 2. \tag{5.3}$$

Choosing here $n = m = 1$ (recall that $\chi_{1,1}^{2,3}(q) = 1$) and using the sum side of the Rogers–Ramanujan identities (3.2), we obtain the following formulae:

$$\chi_{1,5}^{3,10}(q) = \chi_{1,1}^{2,5}(q) \chi_{1,2}^{2,5}(q) = q^{\frac{1}{6}} \sum_{\vec{m}=0}^\infty \frac{q^{m_1^2 + m_2^2 + m_1}}{(q)_{m_1} (q)_{m_2}} \tag{5.4}$$

$$\chi_{1,2k-1}^{3,10}(q) + \chi_{1,11-2k}^{3,10}(q) = \chi_{1,k}^{2,5}(q)\chi_{1,k}^{2,5}(q) = q^{\frac{23-12k}{30}} \sum_{\vec{m}=0}^{\infty} \frac{q^{m_1^2+m_2^2+(4-2k)m_1}}{(q)_{m_1}(q)_{m_2}} \tag{5.5}$$

where $k = 1, 2$. These reducible fermionic representations correspond to the choice $\alpha = 0$ for the matrix A given by (4.27).

Using in a similar way the following identities for $k = 1, 2$ (they were found in [18] and generalized to the form similar to (5.2), (5.3) in [3]):

$$\chi_{2,k}^{4,5}(q) = \chi_{1,2}^{3,4}(q)(\chi_{1,(3-k)}^{6,5}(q) - \chi_{1,(2+k)}^{6,5}(q)) \tag{5.6}$$

$$\chi_{1,k}^{4,5}(q) \pm \chi_{3,k}^{4,5}(q) = (\chi_{1,1}^{3,4}(q) \pm \chi_{1,3}^{3,4}(q))(\chi_{2,(3-k)}^{6,5}(q) \mp \chi_{2,(2+k)}^{6,5}(q)) \tag{5.7}$$

and employing any of the fermionic representations for $\mathcal{M}(3, 4)$ and $\mathcal{M}(5, 6)$ discussed above, we get different reducible fermionic representations for $\mathcal{M}(4, 5)$. For instance, substituting (2.11) and (2.16) into (5.6) we obtain

$$\chi_{2,k}^{4,5}(q) = q^{n_{2,k}^{4,5}} \sum_{\vec{m}=0}^{\infty} \frac{(-1)^{m_3} q^{\frac{1}{2}(m_1^2+m_2^2+m_3^2)+m_2m_3+\frac{1}{2}m_1+(2-k)(m_2+m_3)}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}} \quad k = 1, 2. \tag{5.8}$$

One more possibility to extend the list of fermionic representations is to consider relations between characters with rescaled argument q . For instance, we have [3]

$$\chi_{n,2}^{5,6}(q) + \chi_{n,4}^{5,6}(q) = \chi_{1,n}^{2,5}(q^{\frac{1}{2}}) \quad \chi_{n,1}^{5,6}(q) - \chi_{n,5}^{5,6}(q) = \chi_{1,3-n}^{2,5}(q^2) \quad n = 1, 2. \tag{5.9}$$

Together with (3.2) this gives yet another fermionic representation for $\mathcal{M}(5, 6)$

$$\chi_{n,2}^{5,6}(q) + \chi_{n,4}^{5,6}(q) = \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m^2+(1-\frac{n}{2})m}}{(q^{\frac{1}{2}})_m} \quad \chi_{n,1}^{5,6}(q) - \chi_{n,5}^{5,6}(q) = \sum_{m=0}^{\infty} \frac{q^{2m^2+(2n-2)m}}{(q^2)_m}. \tag{5.10}$$

On the other hand, reading equations (5.9) from right to left and using (2.15), (2.16), we obtain alternative sum sides for the Rogers–Ramanujan identities (3.2):

$$\sum_{m=0}^{\infty} \frac{q^{m^2+(2-n)m}}{(q)_m} = \sum_{\vec{m}=0}^{\infty} \frac{q^{(m_1+m_2)^2+m_1+(4-2n)m_2}}{(q^2)_{m_1}(q^2)_{m_2}} = \sum_{\vec{m}=0}^{\infty} \frac{(-1)^{m_2} q^{\frac{1}{4}(m_1+m_2)^2+(1-\frac{n}{2})(m_1+m_2)}}{(q^{\frac{1}{2}})_{m_1}(q^{\frac{1}{2}})_{m_2}} \tag{5.11}$$

where $n = 1, 2$. This sequence of identities can be continued further by employing the fermionic representations (2.17), (2.18) and also those found in [4] (corresponding to (4.27) with $\alpha = \frac{1}{3}$).

Another set of relations observed in [3]

$$\chi_{2,n}^{6,7}(q^2) + \chi_{4,n}^{6,7}(q^2) = \chi_{1,d_n}^{6,7}(q) - \chi_{5,d_n}^{6,7}(q) \tag{5.12}$$

where $d_1 = 3, d_2 = 1, d_3 = 2$, together with equations (2.21), (2.22) can be used to get fermionic representations of the type (4.3) for the $\mathcal{M}(6, 7)$ model.

It is also possible to combine the rescaling of q and the construction of reducible fermionic sums. For example, let p be a prime number such that $\langle p, t \rangle = 1$. Then, using (1.1) and (3.3), it is easy to derive the following relation:

$$\chi_{1,p}^{3,2p}(q)\chi_{n,m}^{s,t}(q^p) = \chi_{pn,m}^{ps,t}(q). \tag{5.13}$$

Here $\chi_{1,p}^{3,2p}(q)$ should be replaced with $\chi_{1,3}^{2,9}(q)$ if $p = 3$. If p is not a prime number, generalization of (5.13) can be achieved by decomposing p into proper factors. Now choosing $p = 2, t = 5$ and $s = 2$ or $s = 3$ in (5.13), we get for $k = 1, 2$

$$\chi_{2,k}^{4,5}(q) = \chi_{1,k}^{2,5}(q^2)\chi_{1,2}^{3,4}(q) \quad \chi_{k,2}^{5,6}(q) \pm \chi_{k,4}^{5,6}(q) = (\chi_{1,k}^{3,5}(q^2) \pm \chi_{1,5-k}^{3,5}(q^2))\chi_{1,2}^{3,4}(q). \tag{5.14}$$

Substituting here (2.20), (2.11) and (3.12), we obtain more fermionic representations of the type (4.3) for the $\mathcal{M}(4, 5)$ and $\mathcal{M}(5, 6)$ models:

$$\chi_{2,k}^{4,5}(q) = \sum_{\bar{m}=0}^{\infty} \frac{q^{2m_1^2 + \frac{1}{2}m_2^2 + (4-2k)m_1 + \frac{1}{2}m_2}}{(q^2)_{m_1}(q)_{m_2}} \quad k = 1, 2 \quad (5.15)$$

$$\chi_{k,2}^{5,6}(q) \pm \chi_{k,4}^{5,6}(q) = \sum_{\bar{m}=0}^{\infty} \frac{(\pm 1)^{m_1} q^{\frac{1}{2}(m_1^2 + m_2^2) + (2-k)m_1 + \frac{1}{2}m_2}}{(q^2)_{m_1}(q)_{m_2}} \quad k = 1, 2. \quad (5.16)$$

6. Discussion

Having a character (linear combination of characters) in the fermionic form (4.3), we can rewrite it as a series $\chi(q) = \sum_{k=0}^{\infty} \mu_k q^k$, where the level k admits partitioning, $k = \sum_{a=1}^r \sum_{i_a} p_a^{i_a}$, into parts of a specific form. The interpretation of the $p_a^{i_a}$ as momenta of massless particles gives rise to the quasi-particle picture, where a character is regarded as a partition function, $\chi(q) = \sum_k \mu_k e^{-\beta E_k}$. Here $q = e^{-2\pi\beta v/L}$, with v being the speed of sound, and L the size of the system. This quasi-particle representation was developed originally in [4] (for $b_i = 1$) and has become a standard technique. It is also applicable to factorized characters [3, 5, 19] (in this case $b_i \neq 1$).

For the fermionic form (1.5) or (4.3) of a character the quasi-particle representation involves r quasi-particles. They are naturally interpreted as a conformal limit of particles presented in a massive theory related to the given conformal model. Moreover, it was suggested in [20] that different non-equivalent fermionic representations of the same character correspond to different integrable perturbations of the conformal model in question. It would be interesting to understand if the representations for $\mathcal{M}(4, 5)$, $\mathcal{M}(5, 6)$ and $\mathcal{M}(6, 7)$ discussed in section 2 agree with this picture. Our results demonstrate that the number of non-equivalent fermionic representations for $\mathcal{M}(s, t)$ increases if st can be represented as a product of two other co-prime numbers. For instance, we have encountered above three representations of the type (1.5) for $\mathcal{M}(5, 6)$ besides the one considered in [4]. Furthermore, we can considerably expand the list of non-equivalent fermionic representations, if we are looking for representations of the type (4.3), including those that are reducible. For instance, in this way one obtains representations with one (5.10) and two quasi-particles (5.16) for $\mathcal{M}(5, 6)$ (another two-particle representation of the type (4.3) follows from the factorized characters [3, 19]).

To summarize, we have extended the list of fermionic representations for some minimal Virasoro models. The physical content of these representations, in particular, their connection with massive integrable models remains to be investigated. For all considered fermionic representations we have established Rogers–Ramanujan-type identities and the corresponding dilogarithm identities. The Rogers–Ramanujan-type identities can possibly be employed to construct various quasi-particle representations for certain physical entities arising in the lattice models of statistical mechanics.

After submission of this manuscript I was informed by the referee that connection of the matrix (2.12) to the $\mathcal{M}(4, 5)$ model was noticed earlier by M Terhoeven (unpublished).

Acknowledgment

I am grateful to A Fring for useful discussions.

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